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ARTICLE

A family of estimators of finite population mean with dual use of auxiliary information in sample surveys

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Abstract
In this paper we have suggested a family of estimators for estimating the population mean of the study variable with the dual use of auxiliary information in sample surveys. In addition to Haq *et al.* (2017) many other estimators are members of the proposed family of estimators. The bias and mean squared error of the suggested family of estimators are obtained up to the first order of approximation. We have compared the proposed family of estimators with some existing estimators and derived the conditions under which the suggested family of estimators is more efficient than the existing estimators. An empirical study is carried out to demonstrate the performance of the proposed family of estimators over existing estimators.

Keywords: Study Variable; Auxiliary Variable; Bias; Mean Squared Error; Percent Relative Efficiency.

1. Introduction

It is well established fact that the use of auxiliary information at the estimation stage improves the precision of an estimate of the population mean in sample surveys. Ratio, product and regression methods of estimation are good examples in this context. Taking the advantage of high (positive/negative) correlation between the study variable and auxiliary variable, these methods have been defined. Various authors including Sisodia and Dwivedi (1981), Upadhyaya and Singh (1999), Singh and Tailor (2003), Singh and Yadav (2018), Singh *et al.* (2016), Gupta and Shabbir (2008), Kadilar and Cingi (2004, 2006 a,b), Haq and Shabbir (2013), Singh and Solanki (2013), Grover and Kaur (2014), Hussain and Haq (2019), Singh and Nigam (2020, 2022), Kumar *et al.* (2023), Nigam and Singh (2024) and references cited therein have used transformation over auxiliary variable in devising different types of estimators of population mean. Adichwal *et al.* (2019) considered the problem of estimating the population mean using one or two auxiliary variables in the presence of non-response under two phase sampling when study variable is qualitative in nature. Raghav *et al.* (2014, 2023) formulated a multivariate stratified sampling problem in the case of non-response and as a mathematical programming problem to estimate p-population means with complete response and nonresponse for a fixed cost respectively. Recently Mishra *et al.* (2024) introduces a novel class of estimators, combining the ratio and product forms for estimating finite population mean, Raghav *et al.*

(2024) proposes two classes of robust ratio type estimators of finite population mean and of finite population variance using a single auxiliary variable under the adaptive cluster sampling. It is noticed that usually estimators for population mean use auxiliary information on the population mean of the auxiliary variable. Haq *et al.* (2017) pointed out that it is possible to enhance the precision of a mean estimator utilizing ancillary information not only on the auxiliary variable but well on the ranks of the auxiliary variable. This led authors to suggest estimators of the finite population mean that incorporates the ancillary information in forms of: (i) the auxiliary variable and (ii) ranks of the auxiliary variable. The properties of the suggested family of estimators are studied up to first order of approximation. Optimum conditions are obtained under which the suggested estimator has least mean squared error (*MSE*). To evaluate the performance of the proposed class of estimators, a numerical study is carried out using finite real-life population.

Notations

Consider a finite population $\Omega = \{\Omega_1, \Omega_2, ..., \Omega_N\}$ of size *N*. For estimating the population mean, a simple random sample (*SRS*) of size *n* is drawn from Ω using without replacement (*WOR*) sampling scheme. Let (y_i, x_i) be the values of the (study, auxiliary) variables (*Y*,*X*) respectively for the i^{th} unit of the population $\Omega_i, i = 1,2,..., N$. To estimate the population mean $\bar{Y}$ of the study variable *Y*, we suppose that the complete information on the auxiliary variable *X* is available or measurable at an affordable cost, e.g., ranks of the auxiliary variable, and the population parameters such as population mean $\bar{X}$, population standard deviation (S_x), population variance (S_x^2), coefficient of variation (C_x), coefficient of skewness $(\beta_{1(x)})$ and kurtosis $(\beta_{2(x)})$, $\Delta = (\beta_{2(x)} - \beta_{1(x)} - 1)$, etc. of the auxiliary variable *X* and also the correlation coefficient ρ_{yx} between the study variable *Y* and the auxiliary variable *X*. Further we designate:

$\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$: the population mean of the study variable *Y*,

$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$: the population mean of the auxiliary variable *X*,

$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$: the population mean square/variance of *Y*,

$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2$: the population mean square/variance of *X*,

$S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X})$: the population covariance between *Y* and *X*,

$\rho_{yx} = \frac{S_{yx}}{S_y S_x}$: the population correlation coefficient between *Y* and *X*,

$C_y = \frac{S_y}{\bar{Y}}$: the population coefficient of variation of *Y*,

$C_x = \frac{S_x}{\bar{X}}$: the population coefficient of variation of *X*.

Let $x_1, x_2, ..., x_N$ be the *N* values of *X* in the underlying finite population Ω. Let R_x be the corresponding ranks of *X*.

$\bar{R}_x = \frac{1}{N}\sum_{i=1}^{N} r_{x_i} = \frac{(N+1)}{2}$: the population mean of R_x,

$S_{rx}^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(r_{x_i} - \bar{R}_x\right)^2$: the population mean square/variance of R_x,

$S_{zr_x} = \frac{1}{N-1}\sum_{i=1}^{N}\left(z_i - \bar{Z}\right)\left(r_{x_i} - \bar{R}_x\right)$: the population covariance between *Z* and R_x, where $z = y, x$:

$\rho_{zr_x} = \frac{S_{zr_x}}{S_z S_{r_x}}$: the population correlation coefficient between *Z* and R_x,

$C_r = \frac{S_{r_x}}{\bar{R}_x}$: the population coefficient of variation of R_x,

$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$: the sample mean of *Y*,

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$: the sample mean of *X*,

$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$: the sample mean square/variance of *Y*,

$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$: the sample mean square/variance of *X*,

$s_{yx} = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)\left(x_i - \bar{x}\right)$: the population covariance between *Y* and *X*,

For obtaining the bias and mean squared error (*MSE*). We write

$\bar{y} = \bar{Y}(1+e_0), \bar{x} = \bar{X}(1+e_1), \bar{r}_x = \bar{R}_x(1+e_2)$

such that

$E(e_0) = E(e_1) = E(e_2) = 0$

and

$E(e_0^2) = \varphi C_y^2,\ E(e_1^2) = \varphi C_x^2,\ E(e_2^2) = \varphi C_r^2,$

$E(e_0 e_1) = \varphi\rho_{yx} C_y C_x,\ E(e_0 e_2) = \varphi\rho_{yr_x} C_y C_r, E(e_1 e_2) = \varphi\rho_{xr_x} C_x C_r$

and $\phi = \left(\frac{1}{n} - \frac{1}{N}\right)$.

2. Review of Some Existing Estimators

Under this heading we briefly review some existing estimators of population mean $\bar{Y}$ of *Y*. The usual unbiased estimator for $\bar{Y}$ is defined by

$$t_0 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i ,$$

whose variance/*MSE* of $t_0 = \bar{y}$ under *SRSWOR* sampling scheme is given by

$$MSE(t_0) = MSE(\bar{y}) = \varphi S_y^2 = \varphi \bar{Y}^2 C_y^2 \tag{2.1}$$

Ratio estimator due to Cochran (1940) and product estimator due to Robson (1957) and revisited by Murthy (1964) are respectively given by

$$t_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right),$$

and

$$t_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right).$$

To the first degree of approximation, the mean squared errors (*MSE*s) of t_R and t_P are respectively given by

$$MSE(t_R) = \varphi \bar{Y}^2\left(C_y^2 + C_x^2 - 2\rho_{yx} C_y C_x\right) \tag{2.2}$$

$$MSE(t_P) = \varphi \bar{Y}^2\left(C_y^2 + C_x^2 + 2\rho_{yx} C_y C_x\right) \tag{2.3}$$

The usual difference estimator for population mean $\bar{Y}$ of y is defined by

$$t_d = \bar{y} + d\left(\bar{X} - \bar{x}\right)$$

where d is a constant.

For the optimum value $d_{(opt)} = \rho_{yx}\left(\frac{S_y}{S_x}\right)$ of d, the minimum *MSE* of t_d is given by

$$MSE_{\min}(t_d) = \varphi \bar{Y}^2 C_y^2\left(1 - \rho_{yx}^2\right). \tag{2.4}$$

From (2.1), (2.2), (2.3) and (2.4) we note that

$$MSE(\bar{y}) - MSE_{\min}(t_d) = \varphi \bar{Y}^2 C_y^2 \rho_{yx}^2 \geq 0,$$

$$MSE(t_R) - MSE_{\min}(t_d) = \varphi \bar{Y}^2\left(C_x - \rho_{yx} C_y\right)^2 \geq 0,$$

$$MSE(t_P) - MSE_{\min}(t_d) = \varphi \bar{Y}^2\left(C_x + \rho_{yx} C_y\right)^2 \geq 0.$$

It follows from the above three expressions that the difference estimator t_d is superior to the estimators $\bar{y}, t_R$ and t_P.

Rao (1991) envisaged an estimator for $\bar{Y}$ as

$$t_{Rao} = \alpha_1 \bar{y} + \alpha_2\left(\bar{X} - \bar{x}\right),$$

where (α_1, α_2) are suitable chosen constants.

For the optimum values:

$$\alpha_{1(opt)} = \left[1 + \varphi C_y^2\left(1 - \rho_{yx}^2\right)\right]^{-1},$$

and

$$\alpha_{2(opt)} = R\left(\frac{C_y}{C_x}\right)\rho_{yx}\left[1 + \varphi C_y^2\left(1 - \rho_{yx}^2\right)\right]^{-1},$$

of (α_1, α_2) respectively; the minimum *MSE* of t_{Rao} is given by

$$MSE(t_{Rao})_{min.} = \frac{\phi \bar{Y}^2 C_y^2 (1 - \rho_{yx}^2)}{[1 + \phi C_y^2 (1 - \rho_{yx}^2)]} \tag{2.5}$$

From (2.4) and (2.5) we have

$$MSE_{\min}(t_d)-MSE_{\min}(t_{Rao})=\frac{\varphi^2\bar{Y}^2C_y^4\left(1-\rho_{yx}^2\right)^2}{\left[1+\varphi C_y^2\left(1-\rho_{yx}^2\right)\right]}\geq 0$$

which shows that the estimator t_{Rao} due to Rao (1991) is more efficient than the difference t_d and hence more efficient than the estimators $\bar{y}, t_R$ and t_P.

Ratio-type and product-type exponential estimators for $\bar{Y}$ due to Bahl and Tuteja (1991) are given by

$$t_{\mathrm{Re}}=\bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right),$$

$$t_{Pe}=\bar{y}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}}\right).$$

To the first degree of approximation, the *MSE*s of t_{Re} and t_{Pe} are respectively given by

$$MSE(t_{\mathrm{Re}})=\frac{\varphi\bar{Y}^2}{4}\left(4C_y^2+C_x^2-4\rho_{yx}C_yC_x\right) \tag{2.6}$$

$$MSE(t_{Pe})=\frac{\varphi\bar{Y}^2}{4}\left(4C_y^2+C_x^2+4\rho_{yx}C_yC_x\right) \tag{2.7}$$

Singh *et al.* (2009) proposed a class of estimators for $\bar{Y}$

$$t_s=\bar{y}\exp\left\{\frac{a\left(\bar{X}-\bar{x}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\}$$

The minimum *MSE* of t_s to the first degree of approximation is given by

$$MSE_{\min}(t_s)=\varphi\bar{Y}^2C_y^2\left(1-\rho_{yx}^2\right) \tag{2.8}$$

Based on Bedi's (1996) transformation, Haq *et al.* (2017) proposed the following estimator for $\bar{Y}$ as

$$t_{Hq}=\bar{y}\exp\left\{\frac{\left(\bar{X}-\bar{x}\right)}{\bar{X}+\bar{x}+2N\bar{X}}\right\}$$

To the first degree of approximation, the *MSE* of t_H is given by

$$MSE(t_{Hq})=\varphi\bar{Y}^2\left[C_y^2+\frac{C_x^2}{4(N+1)^2}-\frac{1}{(N+1)}\rho_{yx}C_yC_x\right] \tag{2.9}$$

Shabbir and Gupta (2010) envisaged a difference-ratio-type exponential estimator for $\bar{Y}$ as

$$t_{SG}=\left\{w_1\bar{y}+w_2\left(\bar{X}-\bar{x}\right)\right\}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2N\bar{X}}\right),$$

where (w_1,w_2) are suitable chosen constants.

For the optimum values:

$$w_{1(opt)}=\frac{\left(a_2^*a_4^*-a_3^*a_5^*\right)}{a_1^*a_2^*-a_3^{*2}},\ w_{2(opt)}=\frac{\left(a_1^*a_5^*-a_3^*a_4^*\right)}{a_1^*a_2^*-a_3^{*2}}$$

of (w_1,w_2), the minimum *MSE* of t_{SG} is given by

$$MSE_{\min}(t_{SG})=\bar{Y}^2\left[1-\frac{\left(a_2^*a_4^{*2}-2a_3^*a_4^*a_5^*+a_1^*a_5^*\right)}{\left(a_1^*a_2^*-a_3^{*2}\right)}\right], \tag{2.10}$$

where

$$a_1^* = \left[1 + \varphi\left(C_y^2 - 2\theta_1\rho_{yx}C_yC_x + \theta_1^2C_x^2\right)\right],$$

$$a_2^* = \frac{\varphi}{R^2}C_x^2,$$

$$a_3^* = \frac{\varphi}{R}\left(\theta_1C_x^2 - \rho_{yx}C_yC_x\right),$$

$$a_4^* = \left[1 + \frac{\varphi\theta_1}{8}\left(3\theta_1C_x^2 - 4\rho_{yx}C_yC_x\right)\right],$$

$$a_5^* = \frac{\varphi}{2R}C_x^2,$$

$$R = \frac{\bar{Y}}{\bar{X}}, \theta_1 = \frac{1}{N+1}.$$

Grover and Kaur (2011) proposed that a class of estimators for $\bar{Y}$ as

$$t_{GK} = \left[w_1\bar{y} + w_2\left(\bar{X} - \bar{x}\right)\right]\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right),$$

where (w_1, w_2) are suitable chosen constants.

For the optimum values:

$$w_{1(opt)} = \frac{\left(a_{2(1)}a_{4(1)} - a_{3(1)}a_{5(1)}\right)}{\left(a_{1(1)}a_{2(1)} - a_{3(1)}^2\right)}, w_{2(opt)} = \frac{\left(a_{1(1)}a_{5(1)} - a_{3(1)}a_{4(1)}\right)}{\left(a_{1(1)}a_{2(1)} - a_{3(1)}^2\right)},$$

of (w_1, w_2), the minimum *MSE* of t_{GK} is given by

$$MSE_{\min}(t_{GK}) = \bar{Y}^2\left[1 - \frac{\left(a_{2(1)}a_{4(1)}^2 - 2a_{3(1)}a_{4(1)}a_{5(1)} + a_{1(1)}a_{5(1)}^2\right)}{\left(a_{1(1)}a_{2(1)} - a_{3(1)}^2\right)}\right], \tag{2.11}$$

Where

$$a_{1(1)} = \left[1 + \varphi\left(C_y^2 - 2\rho_{yx}C_yC_x + C_x^2\right)\right],$$

$$a_{2(1)} = \frac{\varphi}{R^2}C_x^2,$$

$$a_{3(1)} = \frac{\varphi}{R}\left(C_x^2 - \rho_{yx}C_yC_x\right),$$

$$a_{4(1)} = \left[1 + \frac{\varphi}{8}\left(3C_x^2 - 4\rho_{yx}C_yC_x\right)\right],$$

$$a_{5(1)} = \frac{\varphi}{2R}C_x^2.$$

A generalized version of t_{GK} due to Grover and Kaur (2014) is given by

$$t_{GK}^{(g)} = \left[w_1\bar{y} + w_2\left(\bar{X} - \bar{x}\right)\right]\exp\left\{\frac{a\left(\bar{X} - \bar{x}\right)}{a\left(\bar{X} + \bar{x}\right) + 2b}\right\},$$

where (w_1, w_2) are suitable chosen constants.

The minimum *MSE* of $t_{GK}^{(g)}$ for the optimum values:

$$w_{1(opt)} = \frac{(a_2a_4 - a_3a_5)}{(a_1a_2 - a_3^2)}, w_{2(opt)} = \frac{(a_1a_5 - a_3a_4)}{(a_1a_2 - a_3^2)},$$

of (w_1, w_2) is given by

$$MSE_{\min}\left(t_{GK}^{(g)}\right) = \bar{Y}^2\left[1 - \frac{(a_2a_4^2 - 2a_3a_4a_5 + a_1a_5^2)}{(a_1a_2 - a_3^2)}\right], \tag{2.12}$$

where

$a_1 = \left[1 + \varphi\left(C_y^2 - 2\tau\rho_{yx}C_yC_x + \tau^2C_x^2\right)\right],$

$a_2 = \frac{\varphi}{R^2}C_x^2,$

$a_3 = \frac{\varphi}{R}\left(\tau C_x^2 - \rho_{yx}C_yC_x\right),$

$a_4 = \left[1 + \frac{\varphi\tau}{8}\left(3\tau C_x^2 - 4\rho_{yx}C_yC_x\right)\right],$

$a_5 = \frac{\varphi\theta}{2R}C_x^2, \tau = \frac{a\bar{X}}{a\bar{X} + b}.$

If we set w_1 in t_{SG}, then the estimator $t_{GK}^{(g)}$ reduces to the estimator for the population mean $\bar{Y}$ of y as

$$t_{GK}^{(g)*} = \left[\bar{y} + w_2\left(\bar{X} - \bar{x}\right)\right]\exp\left\{\frac{a\left(\bar{X} - \bar{x}\right)}{a\left(\bar{X} + \bar{x}\right) + 2b}\right\}$$

For the optimum value:

$$w_{2(opt)} = \frac{(a_5 - a_3)}{a_2}$$

of w_2, the minimum *MSE* of $t_{GK}^{(g)*}$ is given by

$$MSE_{\min}\left(t_{GK}^{(g)*}\right) = \bar{Y}^2\left[1 + a_1 - 2a_4 - \frac{(a_5 - a_3)^2}{a_2}\right]$$
$$= \varphi\bar{Y}^2C_y^2\left(1 - \rho_{yx}^2\right) \tag{2.13}$$

which equals to the minimum *MSE* of the difference estimator t_d.

From (2.12) and (2.13) we have

$$MSE_{\min}\left(t_{GK}^{(g)*}\right) - MSE_{\min}\left(t_{GK}^{(g)}\right) = \bar{Y}^2\frac{\left[a_2(a_1 - a_4) + a_3(a_5 - a_3)\right]^2}{a_2\left(a_1a_2 - a_3^2\right)} \geq 0 \tag{2.14}$$

which shows that the proposed estimators $t_{GK}^{(g)}$ is more efficient than $t_{GK}^{(g)*}$ and hence $t_{GK}^{(g)}$ is better than the difference estimator t_d.

Incorporating the supplementary information in the form of an auxiliary variable and in the form of a ranked auxiliary variable (i.e. incorporating the auxiliary information on both X and R_x), Haq *et al.* (2017) suggested a difference estimator of the finite population mean $\bar{Y}$ of y as

$t_{pr} = w_1\bar{y} + w_2\left(\bar{X} - \bar{x}\right) + w_3\left(\bar{R}_x - \bar{r}_x\right),$

where w_1, w_2 and w_3 are suitably chosen constants to be determined such that the *MSE* of t_{pr} is minimum.

For the optimum values:

$$w_{1(opt)} = \frac{\Delta_1^*}{\Delta^*},\quad w_{2(opt)} = \frac{\Delta_2^*}{\Delta^*},\quad w_{3(opt)} = \frac{\Delta_3^*}{\Delta^*},$$

of (w_1, w_2, w_3), the minimum *MSE*s of t_{pr} is given by

$$MSE_{\min}(t_{pr}) = \bar{Y}^2\left[1 - \frac{\Delta_1^*}{\Delta^*}\right], \tag{2.15}$$

where

$$\Delta^* = \left[b_1^*\left(b_2^*b_3^* - b_6^{*2}\right) - b_4^*\left(b_3^*b_4^* - b_5^*b_6^*\right) + b_5^*\left(b_4^*b_6^* - b_2^*b_5^*\right)\right]$$

$$\Delta_1^* = \left(b_2^*b_3^* - b_6^{*2}\right),\ \Delta_2^* = \left(b_3^*b_4^* - b_5^*b_6^*\right), \Delta_3^* = \left(b_2^*b_5^* - b_4^*b_6^*\right),$$

$$b_1^* = \left(1 + \varphi C_y^2\right), b_2^* = \frac{\varphi}{R^2}C_x^2, b_3^* = \frac{\varphi}{R^*}C_r^2, b_4^* = \frac{\varphi}{R}\rho_{yx}C_yC_x, b_5^* = \frac{\varphi}{R^*}\rho_{yr_x}C_yC_r, b_6^* = \frac{\varphi}{RR^*}\rho_{xr_x}C_xC_r.$$

Further, Haq *et al.* (2017) suggested a class of estimators for population mean $\bar{Y}$ of the study variable *y* as

$$t_{pr}^* = \left[w_1\bar{y} + w_2\left(\bar{X} - \bar{x}\right) + w_3\left(\bar{R}_x - \bar{r}_x\right)\right]\exp\left\{\frac{a\left(\bar{X} - \bar{x}\right)}{a\left(\bar{X} + \bar{x}\right) + 2b}\right\},$$

where (w_1, w_2, w_3) are suitable chosen constants.

The minimum *MSE* of t_{pr}^* for the optimum values:

$$w_{10} = \frac{\Delta_1^{**}}{\Delta^{**}},\quad w_{20} = \frac{\Delta_2^{**}}{\Delta^{**}},\quad w_{30} = \frac{\Delta_3^{**}}{\Delta^{**}},$$

of (w_1, w_2, w_3) respectively, is given by

$$MSE_{\min}(t_{pr}^*) = \bar{Y}^2\left[1 - \frac{\left(b_7\Delta_1^{**} + b_8\Delta_2^{**} + b_9\Delta_3^{**}\right)}{\Delta^{**}}\right] \tag{2.16}$$

where,

$$\Delta_1^{**} = \left[b_7\left(b_2b_3 - b_6^2\right) - b_4\left(b_3b_8 - b_6b_9\right) + b_5\left(b_6b_8 - b_2b_9\right)\right],$$

$$\Delta_2^{**} = \left[b_1\left(b_3b_8 - b_6b_9\right) - b_7\left(b_3b_4 - b_5b_6\right) + b_5\left(b_4b_9 - b_5b_8\right)\right],$$

$$\Delta_3^{**} = \left[b_1\left(b_2b_9 - b_6b_8\right) - b_4\left(b_4b_9 - b_5b_8\right) + b_7\left(b_4b_6 - b_2b_5\right)\right],$$

$$\Delta^{**} = \left[b_1\left(b_2b_3 - b_6^2\right) - b_4\left(b_3b_4 - b_5b_6\right) + b_5\left(b_4b_6 - b_2b_5\right)\right],$$

where

$$b_1 = \left[1 + \varphi\left(C_y^2 + \tau^2C_x^2 - 2\tau\rho_{yx}C_yC_x\right)\right],$$

$$b_2 = \frac{\varphi}{R^2}C_x^2,$$

$$b_3 = \frac{\varphi}{R^{*2}}C_r^2,$$

$$b_4 = \frac{\varphi}{R}\left(\tau C_x^2 - \rho_{yx}C_yC_x\right),$$

$$b_5 = \frac{\varphi}{R^*}\left(\tau\rho_{xr_x}C_xC_r - \rho_{yr_x}C_yC_r\right),$$

$$b_6 = \frac{\varphi}{RR^*}\rho_{xr_x}C_xC_r,$$

$$b_7 = \left[1+\frac{\varphi\tau}{8}\left(3\tau C_x^2 - 4\rho_{yx}C_yC_x\right)\right],$$

$$b_8 = \frac{\varphi\tau}{2R}C_x^2,$$

$$b_9 = \frac{\varphi\tau}{R^*}\rho_{xr_x}C_xC_r,$$

$$R = \frac{\bar{Y}}{\bar{X}} \text{ and } R^* = \frac{\bar{Y}}{\bar{R}_x}.$$

3. Proposed Class of Estimators

We define a class of estimators for population mean $\bar{Y}$ of y as

$$t_H = \left[w_1\bar{y}\left(\frac{a\bar{x}+b}{a\bar{X}+b}\right)^{\eta}\left(\frac{\bar{r}_x}{\bar{R}_x}\right)^{\delta} + w_2(\bar{X}-\bar{x}) + w_3(\bar{R}_x-\bar{r}_x)\right] exp\left\{\frac{\lambda(\bar{R}_x-\bar{r}_x)}{(\bar{R}_x+\bar{r}_x)}\right\} exp\left\{\frac{\gamma a(\bar{X}-\bar{x})}{a(\bar{X}+\bar{x})+2b}\right\}, \quad (3.1)$$

where $a(\neq 0)$ and b are either known as constants or functions of any known population parameters, such as S_x (standard deviation), C_x (coefficient of variation), $\beta_{1(x)}$ (coefficient of skewness) and $\beta_{2(x)}$ kurtosis, $\Delta = \left(\beta_{2(x)} - \beta_{1(x)} - 1\right)$, and correlation coefficient ρ_{yx} between Y and X etc.

We mention that for $(\eta,\delta,\lambda,\gamma) = (0,0,0,0)$, the proposed estimator t_H reduces to the estimator t_{pr} while for $(\eta,\delta,\lambda,\gamma) = (0,0,0,1)$ it boils to the estimator t_{pr}^* due to Haq *et al.* (2017).

For deriving the bias and *MSE* of t_H, we express t_H in terms of e's we have

$$t_H = \bar{Y}\left[w_1(1+e_0)(1+\tau e_1)^{\eta}(1+e_2)^{\delta} - w_2\frac{1}{R}e_1 - w_3\frac{1}{R^*}e_2\right]\exp\left\{\frac{-\gamma\tau e_1}{2}\left(1+\frac{\tau}{2}e_1\right)^{-1}\right\}\exp\left\{\frac{-\lambda e_2}{2}\left(1+\frac{e_2}{2}\right)^{-1}\right\}$$

Expanding the right hand side of the above expression, multiplying out and neglecting terms of e's having power greater than two, we have

$$t_H = \bar{Y}\left[\begin{array}{l} w_1\left\{1+e_0+\theta\tau(e_1+e_0e_1)+\theta^*(e_2+e_0e_2)+\theta\theta^*\tau e_1e_2+\frac{\theta(\theta-1)}{2}\tau^2e_1^2+\frac{\theta^*(\theta^*-1)}{2}e_2^2\right\} \\ +w_2\frac{1}{R}\left(\frac{\gamma\tau}{2}e_1^2+\frac{\lambda}{2}e_1e_2-e_1\right)+w_3\frac{1}{R^*}\left(\frac{\gamma\tau}{2}e_1e_2+\frac{\lambda}{2}e_2^2-e_2\right)\end{array}\right]$$

or

$$\left(t_H-\bar{Y}\right)\cong\bar{Y}\left[\begin{array}{l} w_1\left\{1+e_0+\theta\tau(e_1+e_0e_1)+\theta^*(e_2+e_0e_2)+\theta\theta^*\tau e_1e_2+\frac{\theta(\theta-1)}{2}\tau^2e_1^2+\frac{\theta^*(\theta^*-1)}{2}e_2^2\right\} \\ +w_2\frac{1}{R}\left(\frac{\gamma\tau}{2}e_1^2+\frac{\lambda}{2}e_1e_2-e_1\right)+w_3\frac{1}{R^*}\left(\frac{\gamma\tau}{2}e_1e_2+\frac{\lambda}{2}e_2^2-e_2\right)-1\end{array}\right], \quad (3.2)$$

where

$$\theta = \frac{1}{2}(2\eta-\gamma), \theta^* = \frac{1}{2}(2\delta-\lambda), \tau = \frac{a\bar{X}}{a\bar{X}+b}, R = \frac{\bar{Y}}{\bar{X}} \text{ and } R^* = \frac{\bar{Y}}{\bar{R}_x}.$$

Taking expectation of both sides of (3.2) we get the bias of t_H to the first degree of approximation as

$$B(t_H)=\bar{Y}\left[w_1A_7+w_2A_8+w_3A_9-1\right], \tag{3.3}$$

where

$$A_7=\left[1+\varphi\left\{\theta\tau\rho_{yx}C_yC_x+\theta^*\rho_{yr_x}C_yC_r+\theta\theta^*\tau\rho_{xr_x}C_xC_r+\frac{\theta(\theta-1)}{2}\tau^2C_x^2+\frac{\theta^*(\theta^*-1)}{2}C_r^2\right\}\right]$$

$$A_8=\frac{\varphi}{2R}\left(\gamma\tau C_x^2+\lambda\rho_{xr_x}C_xC_{r_x}\right),$$

$$A_9=\frac{\varphi}{2R^*}\left(\gamma\tau\rho_{xr_x}C_xC_r+\lambda C_r^2\right),\varphi=\left(\frac{1}{n}-\frac{1}{N}\right).$$

Squaring both sides of (3.2) and neglecting terms of e's having power greater than two, we have

$$\left(t_H-\bar{Y}\right)^2=\bar{Y}^2\left[\begin{array}{l}1+w_1^2\left\{\begin{array}{l}1+2e_0+2\theta\tau e_1+2\theta^*e_2+e_0^2+4\theta\tau e_0e_1+4\theta^*e_0e_2\\+4\theta\theta^*\tau e_1e_2+\theta(2\theta-1)\tau^2e_1^2+\theta^*(2\theta^*-1)e_2^2\end{array}\right\}\\+w_2^2\left(\frac{1}{R^2}\right)e_1^2+w_3^2\left(\frac{1}{R^{*2}}\right)e_2^2+2w_1w_2\left(\frac{1}{R}\right)\left\{(\gamma-\eta)\tau e_1^2+(\lambda-\delta)e_1e_2-e_0e_1-e_1\right\}\\+2w_1w_3\frac{1}{R^*}\left\{(\gamma-\eta)\tau e_1e_2+(\lambda-\delta)e_2^2-e_0e_2-e_2\right\}+2w_2w_3\frac{1}{RR^*}e_1e_2\\-2w_1\left\{1+e_0+\theta\tau(e_1+e_0e_1)+\theta^*(e_2+e_0e_2)+\theta\theta^*\tau e_1e_2+\frac{\theta(\theta-1)}{2}\tau^2e_1^2+\frac{\theta^*(\theta^*-1)}{2}e_2^2\right\}\\-2w_2\frac{1}{R}\left\{\frac{1}{2}\left(\gamma\tau e_1^2+\lambda e_1e_2\right)-e_1\right\}-2w_3\frac{1}{R^*}\left\{\frac{1}{2}\left(\gamma\tau e_1e_2+\lambda e_2^2\right)-e_2\right\}\end{array}\right] \tag{3.4}$$

Taking expectation of both sides of (3.4) we get the *MSE* of t_H to the first degree of approximation as

$$MSE(t_H)=\bar{Y}^2\left[1+w_1^2A_1+w_2^2A_2+w_3^2A_3+2w_1w_2A_4+2w_1w_3A_5+2w_2w_3A_6-2w_1A_7-2w_2A_8-2w_3A_9\right], \tag{3.5}$$

where

$$A_1=\left[1+\varphi\left\{C_y^2+4\theta\tau\rho_{yx}C_yC_x+4\theta^*\rho_{yr_x}C_yC_r+4\theta\theta^*\tau\rho_{xr_x}C_xC_r+\theta(2\theta-1)\tau^2C_x^2+\theta^*(2\theta^*-1)C_r^2\right\}\right],$$

$$A_2=\frac{\varphi}{R^2}C_x^2,$$

$$A_3=\frac{\varphi}{R^{*2}}C_r^2,$$

$$A_4=\frac{\varphi}{R}\left[(\gamma-\eta)\tau C_x^2+(\lambda-\delta)\rho_{xr_x}C_xC_r-\rho_{yx}C_yC_x\right],$$

$$A_5=\frac{\varphi}{R^*}\left[(\gamma-\eta)\tau\rho_{xr_x}C_xC_r+(\lambda-\delta)C_r^2-\rho_{yr_x}C_yC_r\right],$$

$$A_6=\frac{\varphi}{RR^*}\rho_{xr_x}C_xC_r.$$

Setting $\frac{\partial MSE(t_H)}{\partial w_i}=0, i=1,2,3$; we have

$$\begin{bmatrix} A_1 & A_4 & A_5 \\ A_4 & A_2 & A_6 \\ A_5 & A_6 & A_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} A_7 \\ A_8 \\ A_9 \end{bmatrix} \tag{3.6}$$

Solving (3.6) for (w_1, w_2, w_3) we get the optimum values of (w_1, w_2, w_3) respectively as:

$$w_{10} = \frac{\Delta_1}{\Delta}, w_{20} = \frac{\Delta_2}{\Delta}, w_{30} = \frac{\Delta_3}{\Delta};$$

where

$$\Delta = \begin{vmatrix} A_1 & A_4 & A_5 \\ A_4 & A_2 & A_6 \\ A_5 & A_6 & A_3 \end{vmatrix} = \left[A_1\left(A_2A_3 - A_6^2\right) - A_4\left(A_3A_4 - A_5A_6\right) + A_5\left(A_4A_6 - A_2A_5\right)\right]$$

$$\Delta_1 = \begin{vmatrix} A_7 & A_4 & A_5 \\ A_8 & A_2 & A_6 \\ A_9 & A_6 & A_3 \end{vmatrix} = \left[A_7\left(A_2A_3 - A_6^2\right) - A_4\left(A_3A_8 - A_6A_9\right) + A_5\left(A_6A_8 - A_2A_9\right)\right]$$

$$\Delta_2 = \begin{vmatrix} A_1 & A_7 & A_5 \\ A_4 & A_8 & A_6 \\ A_5 & A_9 & A_3 \end{vmatrix} = \left[A_1\left(A_3A_8 - A_6A_9\right) - A_7\left(A_3A_4 - A_5A_6\right) + A_5\left(A_4A_9 - A_5A_8\right)\right]$$

$$\Delta_3 = \begin{vmatrix} A_1 & A_4 & A_7 \\ A_4 & A_2 & A_8 \\ A_5 & A_6 & A_9 \end{vmatrix} = \left[A_1\left(A_2A_9 - A_6A_8\right) - A_4\left(A_4A_9 - A_5A_8\right) + A_7\left(A_4A_6 - A_2A_5\right)\right]$$

Thus the resulting minimum MSE of t_H is given by

$$MSE_{\min}(t_H) = \bar{Y}^2\left[1 - \frac{\left(A_7\Delta_1 + A_8\Delta_2 + A_9\Delta_3\right)}{\Delta}\right] \tag{3.7}$$

which is non-negative if

$$0 < \frac{\left(A_7\Delta_1 + A_8\Delta_2 + A_9\Delta_3\right)}{\Delta} < 1$$

and

$$\Delta > 0.$$

4. Efficiency Comparisons

In this section, we have compared the suggested estimator t_H with the existing competing estimators. Conditions are obtained under which the proposed estimator t_H is better than the existing estimators considered here.

We designate:

$$L_R = \left[1 + \varphi C_y^2\left(1 - \rho_{yx}^2\right)\right]^{-1}, \tag{4.1}$$

$$L_{SG} = \frac{\left(a_2^* a_4^{*2} - 2a_3^* a_4^* a_5^* + a_1^* a_5^{*2}\right)}{\left(a_1^* a_2^* - a_3^{*2}\right)}, \tag{4.2}$$

$$L_{GK} = \frac{\left(a_{2(1)}a_{4(1)}^2 - 2a_{3(1)}a_{4(1)}a_{5(1)} + a_{1(1)}a_{5(1)}^2\right)}{\left(a_{1(1)}a_{2(1)} - a_{3(1)}^2\right)}, \tag{4.3}$$

$$L_{GK}^2 = \frac{\left(a_2a_4^2 - 2a_3a_4a_5 + a_1a_5^2\right)}{\left(a_1a_2 - a_3^2\right)}, \tag{4.4}$$

$$L_P^* = \frac{\Delta_1^*}{\Delta^*}, \tag{4.5}$$

$$L_P^{**} = \frac{b_7\Delta_1^{**} + b_8\Delta_2^{**} + b_9\Delta_3^{**}}{\Delta^{**}}, \tag{4.6}$$

$$L_H = \frac{A_7\Delta_1 + A_8\Delta_2 + A_9\Delta_3}{\Delta^{**}}. \tag{4.7}$$

From (2.1), (2.2), (2.3), (2.4) (or (2.8) or (2.13)), (2.5), (2.6), (2.7), (2.9), (2.10), (2.11), (2.12), (2.15), (2.16) and (2.17) we have that

(i) $MSE_{\min}(t_H) < MSE(t_0 = \bar{y})$ if

$$\left(L_H + \varphi C_y^2\right) > 1 \tag{4.8}$$

(ii) $MSE_{\min}(t_H) < MSE(t_R)$ if

$$\left[L_H + \varphi\left(C_y^2 + C_x^2\right)\right] > \left(1 + 2\varphi\rho_{yx}C_yC_x\right) \tag{4.9}$$

(iii) $MSE_{\min}(L_H) < MSE(t_P)$ if

$$\left[L_H + \varphi\left(C_y^2 + C_x^2 + 2\rho_{yx}C_yC_x\right)\right] > 1 \tag{4.10}$$

(iv) $MSE_{\min}(L_H) < MSE\left(t_d \text{ or } t_s \text{ or } t_{GK}^{(g)*}\right)$ if

$$\left(L_H + \varphi C_y^2\right) > \left(1 + \varphi C_y^2\rho_{yx}^2\right) \tag{4.11}$$

(v) $MSE_{\min}(L_H) < MSE(t_{Rao})$ if

$$L_H > L_R \tag{4.12}$$

(vi) $MSE_{\min}(L_H) < MSE(t_{\mathrm{Re}})$ if

$$\left[L_H + \varphi\left(C_y^2 + \frac{1}{4}C_x^2\right)\right] > \left(1 + \varphi\rho_{yx}C_yC_x\right) \tag{4.13}$$

(vii) $MSE_{\min}(t_H) < MSE(t_{Pe})$ if

$$\left[L_H + \varphi\left(C_y^2 + \frac{1}{4}C_x^2 + \rho_{yx}C_yC_x\right)\right] > 1 \tag{4.14}$$

(viii) $MSE_{\min}(t_H) < MSE(t_{Hq})$ if

$$\left[L_H + \varphi\left(C_y^2 + \frac{1}{4(N+1)^2}C_x^2\right)\right] > \left(1 + \varphi\frac{1}{(N+1)}\rho_{yx}C_yC_x\right) \tag{4.15}$$

(ix) $MSE_{\min}(t_H) < MSE(t_{SG})$ if

$$L_H > L_{SG} \tag{4.16}$$

(x) $MSE_{\min}(t_H) < MSE(t_{GK})$ if

$$L_H > L_{GK} \tag{4.17}$$

(xi) $MSE_{\min}(t_H) < MSE\left(t_{GK}^{(g)}\right)$if

$$L_H > L_{GK}^{*} \tag{4.18}$$

(xii) $MSE_{\min}(t_H) < MSE(t_{pr})$if

$$L_H > L_P^{*} \tag{4.19}$$

(xiii) $MSE_{\min}(t_H) < MSE(t_{pr}^{*})$if

$$L_H > L_P^{**} \tag{4.20}$$

Thus the proposed class of estimators t_H is more efficient than $t_0 = \bar{y}$, t_R , t_P , t_d $\left(\text{or}\, t_s \text{ or}\, t_{GK}^{(g)*}\right)$, t_{Rao} , t_{Re} , t_{Pe} , t_{Hq} , t_{SG} , t_{GK} , $t_{GK}^{(g)}$, t_{pr} and t_{pr}^{*} respectively as long as the conditions (4.8), (4.9), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), (4.17), (4.18), (4.19) and (4.20) are satisfied.

5. Empirical Study

To judge the merits of the proposed class of estimator t_H over other estimators, we have considered a data set reported in Murthy (1967).

Data set [**Source: Murthy, 1967**]
y: Output of factory, and
x: Number of workers.

$N = 80, n = 10, \bar{Y} = 5182.64, \bar{X} = 285.125, \bar{R}_x = 40.50, \rho_{yx} = 0.914981, \rho_{yr_x} = 0.983609,$

$\rho_{xr_x} = 0.890219, C_y = 0.354194, C_x = 0.948459, C_r = 0.573765, \beta_2(x) = 3.58078$

We have computed the percent relative efficiencies (*PRE*s) of different estimators of population mean $\bar{Y}$ with respect to usual unbiased estimator $\bar{y}$. The *PRE*s of the estimators t_R , t_{Re} , t_{Hq} , t_d , t_{Rao} , t_{GK} and t_{pr} (Haq *et al.* (2017) with respect to $\bar{y}$ are displayed in Table 5.1. the *PRE*s of the estimators $t_{GK}^{(g)}$ due to Grover and Kaur (2014) and t_{pr}^{*} due to Haq *et al.* (2017) for various values of (*a*,*b*) are presented in Table 5.2. We have computed the *PRE* of the proposed family of estimators t_H with respect to $\bar{y}$ by using the formula:

$$PRE(t_H, \bar{y}) = \frac{\varphi C_y^2}{\left[1 - \frac{(A_7\Delta_1 + A_8\Delta_2 + A_9\Delta_3)}{\Delta}\right]} * 100$$

For different values of (*a*,*b*) and findings are shown in Table 5.3.

Table 1. *PRE*s of the estimators t_R , t_{Re} , t_{Hq} , t_d , t_{Rao} , t_{GK} and t_{pr} with respect to $\bar{y}$

Estimator	$\bar{y}$	t_R	t_{Re}	t_{Hq}	t_d	t_{Rao}	t_{GK}	t_{pr}
$PRE(.,\bar{y})$	100.00	30.58	291.96	103.9	614.21	615.31	664.37	3437.60

Table 2. *PRE*s of $t_{GK}^{(g)}$ and t_{pr}^{*} with respect to $\bar{y}$ for different values of (a,b)

Values of constants		Members of $t_{GK}^{(g)}$	$PRE\left(t_{GK}^{(g)},\bar{y}\right)$	Members of t_{pr}^{*}	$PRE\left(t_{pr}^{*},\bar{y}\right)$
a	b				
1	C_x	$t_{GK(1)}^{(g)}$	663.77	$t_{pr(1)}^{*}$	6307.63
1	$\beta_2(x)$	$t_{GK(2)}^{(g)}$	662.14	$t_{pr(2)}^{*}$	6182.12
$\beta_2(x)$	C_x	$t_{GK(3)}^{(g)}$	664.20	$t_{pr(3)}^{*}$	6342.04
C_x	$\beta_2(x)$	$t_{GK(4)}^{(g)}$	662.02	$t_{pr(4)}^{*}$	6173.26
1	ρ_{yx}	$t_{GK(5)}^{(g)}$	663.79	$t_{pr(5)}^{*}$	6309.30
C_x	ρ_{yx}	$t_{GK(6)}^{(g)}$	663.76	$t_{pr(6)}^{*}$	6306.82
ρ_{yx}	C_x	$t_{GK(7)}^{(g)}$	663.71	$t_{pr(7)}^{*}$	6303.25
$\beta_2(x)$	ρ_{yx}	$t_{GK(8)}^{(g)}$	664.21	$t_{pr(8)}^{*}$	6342.51
ρ_{yx}	$\beta_2(x)$	$t_{GK(9)}^{(g)}$	661.94	$t_{pr(9)}^{*}$	6167.00
$N\bar{X}$	1	$t_{GK(10)}^{(g)}$	615.31	$t_{pr(10)}^{*}$	6355.75

Table 3. *PRE*s of proposed family of estimators t_H with respect to usual unbiased estimator $\bar{y}$ for different values of $(a,b,\eta,\delta,\lambda,\gamma)$

Values of		$(\eta,\delta,\lambda,\gamma)=(-0.5,0,0,0)$	$(\eta,\delta,\lambda,\gamma)=(0,0,1.5,1.5)$	$(\eta,\delta,\lambda,\gamma)=(0,0,2,1)$
a	b	$PRE\left(t_{H1},\bar{y}\right)$	$PRE\left(t_{H2},\bar{y}\right)$	$PRE\left(t_{H3},\bar{y}\right)$
1	C_x	52516.81	16737.52	8031.265
1	$\beta_2(x)$	32851.4	16022.3	8002.068
$\beta_2(x)$	C_x	62586.32	16935.64	8038.587
C_x	$\beta_2(x)$	31991.54	15972.24	7999.85
1	ρ_{yx}	52931.06	16747.1	8031.626
C_x	ρ_{yx}	52318.24	16732.88	8031.09
ρ_{yx}	C_x	51458	16712.37	8030.313
$\beta_2(x)$	ρ_{yx}	62752.07	16938.39	8038.687
ρ_{yx}	$\beta_2(x)$	31409.93	15936.92	7998.27
$N\bar{X}$	1	3992.193	4100.563	3992.009

*where $t_{H1}=\left[w_1\bar{y}\left\{\frac{\left(a\bar{X}+b\right)}{\left(a\bar{x}+b\right)}\right\}^{\frac{1}{2}}+w_2\left(\bar{X}-\bar{x}\right)+w_3\left(\bar{R}_x-\bar{r}_x\right)\right]$,

$$t_{H2}=\left[w_1\bar{y}+w_2\left(\bar{X}-\bar{x}\right)+w_3\left(\bar{R}_x-\bar{r}_x\right)\right]\exp\left\{\frac{1.5\left(\bar{R}_x-\bar{r}_x\right)}{\left(\bar{R}_x+\bar{r}_x\right)}\right\}\exp\left\{\frac{1.5a\left(\bar{X}-\bar{x}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\},$$

$$t_{H3}=\left[w_1\bar{y}+w_2\left(\bar{X}-\bar{x}\right)+w_3\left(\bar{R}_x-\bar{r}_x\right)\right]\exp\left\{\frac{2\left(\bar{R}_x-\bar{r}_x\right)}{\left(\bar{R}_x+\bar{r}_x\right)}\right\}\exp\left\{\frac{a\left(\bar{X}-\bar{x}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\}.$$

Table 4. *PRE*s of proposed family of estimators t_H with respect to usual unbiased estimator $\bar{y}$ for different values of $(a, b, \eta, \delta, \lambda, \gamma)$

Values of		$(\eta,\delta,\lambda,\gamma)=(0,0,1,1.75)$	$(\eta,\delta,\lambda,\gamma)=(0.5,0.25,0,0)$	$(\eta,\delta,\lambda,\gamma)=(0,-1,0,0)$
a	**b**	$PRE(t_{H4},\bar{y})$	$PRE(t_{H5},\bar{y})$	$PRE(t_{H6},\bar{y})$
1	C_x	7175.57	6879.26	7641.01
1	$\beta_2(x)$	7066.70	6885.67	7641.01
$\beta_2(x)$	C_x	7204.83	6877.33	7641.01
C_x	$\beta_2(x)$	7058.88	6886.08	7641.01
1	ρ_{yx}	7177.00	6879.16	7641.01
C_x	ρ_{yx}	7174.88	6879.30	7641.01
ρ_{yx}	C_x	7171.83	6879.50	7641.01
$\beta_2(x)$	ρ_{yx}	7205.23	6877.30	7641.01
ρ_{yx}	$\beta_2(x)$	7053.35	6886.37	7641.01
$N\bar{X}$	1	4195.09	4061.73	7641.01

*where $t_{H4}=\left[w_1\bar{y}+w_2\left(\bar{X}-\bar{x}\right)+w_3\left(\bar{R}_x-\bar{r}_x\right)\right]\exp\left\{\frac{\bar{R}_x-\bar{r}_x}{\bar{R}_x+\bar{r}_x}\right\}\exp\left\{\frac{1.75a\left(\bar{X}-\bar{x}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\}$

$$t_{H5}=\left[w_1\bar{y}\left\{\frac{\left(a\bar{X}+b\right)}{\left(a\bar{x}+b\right)}\right\}^{\frac{1}{2}}\left(\frac{\bar{r}_x}{\bar{R}_x}\right)^{\frac{1}{4}}+w_2\left(\bar{X}-\bar{x}\right)+w_3\left(\bar{R}_x-\bar{r}_x\right)\right],$$

$$t_{H6}=\left[w_1\bar{y}\left(\frac{\bar{R}_x}{\bar{r}_x}\right)+w_2\left(\bar{X}-\bar{x}\right)+w_3\left(\bar{R}_x-\bar{r}_x\right)\right].$$

Table 5. *PRE*s of proposed family of estimators t_H with respect to usual unbiased estimator $\bar{y}$ for different values of $(a, b, \eta, \delta, \lambda, \gamma)$

Values of		$(\eta,\delta,\lambda,\gamma)=(-0.5,1,1,1)$	$(\eta,\delta,\lambda,\gamma)=(-0.5,1,1,-0.25)$	$(\eta,\delta,\lambda,\gamma)=(-0.25,1,1,-0.25)$
A	**b**	$PRE(t_{H7},\bar{y})$	$PRE(t_{H8},\bar{y})$	$PRE(t_{H9},\bar{y})$
1	C_x	31831.89	10760.63	11212.82
1	$\beta_2(x)$	25105.5	10392.41	11247.21
$\beta_2(x)$	C_x	34309.94	10864.21	11204.37
C_x	$\beta_2(x)$	24731.35	10366.97	11249.87
1	ρ_{yx}	31944.05	10765.62	11212.4
C_x	ρ_{yx}	31777.79	10758.21	11213.02
ρ_{yx}	C_x	31540.76	10747.52	11213.92
$\beta_2(x)$	ρ_{yx}	34346.79	10865.65	11204.26
ρ_{yx}	$\beta_2(x)$	24473.4	10349.05	11251.77
$N\bar{X}$	1	*	*	*

*stands for non-existence of positivity condition at (3.7).

where

$$t_{H7}=\left[w_1\bar{y}\left\{\frac{\left(a\bar{X}+b\right)}{\left(a\bar{x}+b\right)}\right\}^{\frac{1}{2}}\left(\frac{\bar{r}_x}{\bar{R}_x}\right)+w_2\left(\bar{X}-\bar{x}\right)+w_3\left(\bar{R}_x-\bar{r}_x\right)\right]\exp\left\{\frac{\bar{R}_x-\bar{r}_x}{\bar{R}_x+\bar{r}_x}\right\}\exp\left\{\frac{a\left(\bar{X}-\bar{x}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\},$$

$$t_{H8}=\left[w_1\bar{y}\left\{\frac{\left(a\bar{X}+b\right)}{\left(a\bar{x}+b\right)}\right\}^{\frac{1}{2}}\left(\frac{\bar{r}_x}{\bar{R}_x}\right)+w_2\left(\bar{X}-\bar{x}\right)+w_3\left(\bar{R}_x-\bar{r}_x\right)\right]\exp\left\{\frac{\bar{R}_x-\bar{r}_x}{\bar{R}_x+\bar{r}_x}\right\}\exp\left\{\frac{-0.25a\left(\bar{X}-\bar{x}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\},$$

$$t_{H9}=\left[w_1\bar{y}\left\{\frac{\left(a\bar{X}+b\right)}{\left(a\bar{x}+b\right)}\right\}^{\frac{1}{4}}\left(\frac{\bar{r}_x}{\bar{R}_x}\right)+w_2\left(\bar{X}-\bar{x}\right)+w_3\left(\bar{R}_x-\bar{r}_x\right)\right]\exp\left\{\frac{\bar{R}_x-\bar{r}_x}{\bar{R}_x+\bar{r}_x}\right\}\exp\left\{\frac{-0.25a\left(\bar{X}-\bar{x}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\}.$$

It is observed from Tables 5.1 and 5.2 that the estimator $t^*_{pr(8)}$ (i.e. the estimator t^*_{pr} at $(a,b)=(\beta_2(x),\rho_{yx})$) is the best among the estimators $\bar{y}, t_R, t_{\mathrm{Re}}, t_{Hq}, t_d, t_{Rao}, t_{GK}, t_{pr}, t^{(g)}_{GK(j)}, j$ = 1 to 10; and $t^*_{pr(i)}$, i = 1 to 7,9,10; that is the estimator $t^*_{pr(8)}$ has the highest *PRE*(=6342.51%) among these estimators.

Comparing the *PRE* values of Table 5.1, 5.2 and 5.3, it is found that the proposed estimator t_H has *PRE*s considerably larger than the estimators $\bar{y}, t_R, t_{\mathrm{Re}}, t_{Hq}, t_d, t_{Rao}, t_{GK}, t_{pr}, t^{(g)}_{GK(j)}, j$ = 1 to 10; and $t^*_{pr(i)}$, i = 1 to 7,9,10 for the combination of scalars $(a,b,\eta,\delta,\lambda,\gamma)$ considered. The largest *PRE*(=62752.07%) is observed at $(a,b,\eta,\delta,\lambda,\gamma)=(-0.5,0,0,0,\beta_2(x),\rho_{yx})$ which is very high as compared to Haq *et al.* (2017) estimator $t^*_{pr(8)}$. This study suggests that there is enough scope of selecting the scalars $(a,b,\eta,\delta,\lambda,\gamma)$ involved in the suggested estimator t_H for obtaining estimators better than those with their counterparts. Finally, it is recommended that the use of the envisaged estimator t_H for precisely estimating the finite population mean in practice.

6. Conclusion

In this article we have developed a family of estimators of the finite population mean that utilizes the ancillary information on the auxiliary variable as well as on the ranks of the auxiliary variable. In addition to Haq *et al.* (2017) estimators t_{pr} and t^*_{pr}, a large number of estimators can be identified as members of the developed family of estimators t_H. Expressions of bias and mean squared error of the suggested family of estimators t_H have been obtained up to first order of approximation. Optimum conditions are obtained at which the proposed family of estimators t_H attained the minimum mean squared error. Theoretical conditions are obtained under which the proposed family of estimators t_H is more efficient than those with their counterparts. An empirical study is carried out in support of the present study. Findings of the empirical study show the superiority of the envisaged family of estimators t_H over Haq *et al.* (2017) estimators t_{pr} and t^*_{pr}, and other existing estimators. Thus the application of the present study is recommended for its use in practice.

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Conflicts of Interest

The authors declare no conflict of interest.

Author contributions

Conceptualization: NIGAM, P, SINGH, H.P.; **Data curation:** NIGAM, P, SINGH, H.P.; **Formal analysis:** NIGAM, P, SINGH, H.P.; **Funding acquisition:** NIGAM, P, SINGH, H.P.; **Investigation:** NIGAM, P, SINGH, H.P.; **Methodology:** NIGAM, P, SINGH, H.P.; **Project administration:** NIGAM, P, SINGH, H.P.; **Software:** NIGAM, P, SINGH, H.P.; **Resources:** NIGAM, P, SINGH, H.P.; **Supervision:** NIGAM, P, SINGH, H.P.; **Validation:** NIGAM, P, SINGH, H.P.; **Visualization:** NIGAM, P, SINGH, H.P.; **Writing - original draft:** NIGAM, P, SINGH, H.P.; **Writing - review and editing:** NIGAM, P, SINGH, H.P.

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